Matematisk-fysiske Meddelelser
udgivet af
Det Kongelige Danske Videnskabernes Selskab
Bind **33,** nr. 1

Mat. Fys. Medd. Dan. Vid. Selsk. **33,** no. 1 (1960)

THE LOGARITHMIC POTENTIAL IN HIGHER DIMENSIONS

BY

BENT FUGLEDE

København 1960
i kommission hos Ejnar Munksgaard

Synopsis

Various classical potential theoretic properties of the logarithmic kernel in the plane are extended to the logarithmic kernel $-\log|x-y|$ in Euclidean n-space R^n. The key result is the following inequality for the energy of any (signed) mass distribution μ on a ball $B \subset R^n$ of radius ϱ:

$$-\int_B \int_B \log|x-y| \, d\mu(x) \, d\mu(y) \geq \log \frac{a_n}{\varrho} \cdot \left(\int_B d\mu\right)^2.$$

The best possible value of the constant a_n is determined explicitly in its dependence on the dimension n. In particular, the logarithmic kernel satisfies the energy principle on any ball of radius $\varrho < a_n$.

Printed in Denmark
Bianco Lunos Bogtrykkeri A-S

1. Introduction

In view of its role in the theory of analytic or harmonic functions, the logarithmic potential in the plane has been investigated thoroughly. Restricting the attention to the more recent literature on this subject and to the potential theoretic aspects thereof, we mention the works of O. Frostman [5], [6], M. Riesz [12], Ch. de la Vallée-Poussin [14], H. Cartan [1], and G. Choquet [3]. On the other hand, very little research seems to have been devoted to the logarithmic potential in Euclidean space R^n of dimension $n > 2$. The principle results on this topic are those of M. Riesz [12, § 4] and O. Frostman [6, § 1] concerning the logarithmic potential and energy of distributions of algebraic total mass zero, and further the calculation of the Fourier-Schwartz transform of the logarithmic kernel, cf. L. Schwartz [13, ch. VII, § 7] or J. Deny [4, note 3, p. 160 f.].

In the present paper we continue the study of the logarithmic kernel in R^n for arbitrary dimension n; that is, the kernel

$$-\log|x-y| \qquad (x \in R^n,\ y \in R^n),$$

interpreted as $+\infty$ for $x = y$. We shall use the terminology[1] and some of the results of a previous memoir [7]. Most of the results of the present paper are applied in a recent article [8]. Of independent interest is the main result asserting that the logarithmic kernel is *strictly* (positive) *definite* (that is, it satisfies the energy principle) when considered on a ball $A \subset R^n$ of sufficiently small radius a (cf. de la Vallée-Poussin [14, § 47] for the case $n = 2$). The least upper bound a_n of such radii is determined explicitly (§ 4, formula (5)). The proof is based on an explicit computation of the *equilibrium distribution* (in the sense of Deny [4, § 5]) on the unit ball in R^n. Combining this result with the known fact that the logarithmic kernel

[1] Observe, however, that the notations $k(x, \mu)$ and $k(\mu, \nu)$ for potential and mutual energy in [7] will be replaced by $U^{\mu}_{\alpha}(x)$ and $\langle \mu, \nu \rangle$, respectively, in the present paper (in which $k(x, y) = -\log|x-y|$).

is *regular* (that is, it satisfies the continuity principle), it follows, in essence from a theorem of M. OHTSUKA [10], that the logarithmic kernel is *perfect* in the sense of [7, § 3.3] when considered on a ball of radius $a \underset{(=)}{<} a_n$. This is Theorem 4.1 of the present paper. Using [7], one derives various corollaries from this theorem, in particular the existence of an interior or exterior capacitary distribution associated with any given bounded set, and further the capacitability of all bounded analytic subsets of R^n. This last result, which depends strongly on Choquet's theory of capacitability [2], was known previously for $n = 2$ (cf. CHOQUET [3], whose proof is based on special properties of the logarithmic potential in the plane). Further results involving the logarithmic potential or the logarithmic capacity are obtained in [8].

Since the logarithmic kernel is of variable sign, we shall consider the logarithmic potential and energy only of distributions of *compact support*. This limitation will not always be repeated. We shall mainly deal with distributions which are *measures* (not necessarily positive), but general distributions in the sense of SCHWARTZ [13] will enter in the proof of the key result (Lemma 4.1).

2. Basic notions connected with the logarithmic kernel

The logarithmic *potential* of a measure μ on R^n (of compact support) is defined by

$$U^{\mu}(x) = -\int \log |x-y| \, d\mu(y) = U^{\mu^+}(x) - U^{\mu^-}(x)$$

at any point x for which the third member is defined (i. e., not of the form $(+\infty)-(+\infty)$). In particular, $U^{\mu}(x)$ is always defined and $\neq -\infty$ if $\mu \geq 0$.

The logarithmic *mutual energy* $\langle \mu, \nu \rangle$ of two measures μ and ν (both of compact support) is defined by

$$\begin{aligned} \langle \mu, \nu \rangle &= -\iint \log |x-y| \, d\mu(x) d\nu(y) \\ &= \langle \mu^+, \nu^+ \rangle + \langle \mu^-, \nu^- \rangle - \langle \mu^+, \nu^- \rangle - \langle \mu^-, \nu^+ \rangle, \end{aligned}$$

provided the third member is meaningful. In particular, $\langle \mu, \nu \rangle$ is always defined and $\neq -\infty$ if $\mu \geq 0$, $\nu \geq 0$. For $\mu = \nu$ we obtain the logarithmic *energy* $\langle \mu, \mu \rangle$ of a measure μ. An application of Fubini's theorem leads to the *formula of reciprocity*

$$\langle \mu, \nu \rangle = \int U^{\mu}(x)\, d\nu = \int U^{\nu}(x)\, d\mu,$$

valid whenever $\langle \mu, \nu \rangle$ is defined (cf. [7, § 2.1] for details).

In several respects the logarithmic kernel may be viewed as a limit case of the kernels of order α, $|x-y|^{\alpha-n}$, as $\alpha \to n$. This appears, e.g., from the identity

$$\log |x| = \{(\partial/\partial\alpha)\, |x|^{\alpha-n}\}_{\alpha=n}. \tag{1}$$

As observed by M. Riesz [12, § 4], the analogy is almost perfect when the measure μ in question has algebraic total mass 0, $\int d\mu = 0$. Note, in particular, the following formula due to M. Riesz (cf. also Frostman [5, § 33] and [6, § 1]):

$$\langle \mu, \mu \rangle = \frac{1}{\omega_n} \int [U^{\mu}_{n/2}]^2\, dx\, (\geqq 0) \quad \text{if} \quad \int d\mu = 0 \tag{2}$$

under the additional assumption that $\langle \mu, \mu \rangle$ exists and is finite.[1] Here

$$\omega_n = 2\pi^{n/2}/\Gamma(n/2)$$

denotes the surface of the unit sphere in R^n.

The *interior logarithmic capacity* $\gamma_*(E)$ of an arbitrary bounded set $E \subset R^n$ is defined by

$$-\log \gamma_*(E) = w(E);\ \text{i. e.,}\ \gamma_*(E) = \exp(-w(E)). \tag{3}$$

Here

$$w(E) = \inf \langle \mu, \mu \rangle \tag{4}$$

as μ ranges over the class of all positive measures of compact support contained in E and of total mass $\int d\mu = 1$. Cf. [7, § 2.3].

If E is compact, this infimum (4) is an actual minimum (cf. [7, Theorem 2.3]), attained by precisely one competing measure λ called the *capacitary distribution of unit mass* on E. (The uniqueness follows from (2) as explained in Remark 2 to Theorem 2.4 in [7], because the difference $\mu - \nu$ between any two competing measures is of zero total mass. Moreover, $U^{\mu}_{n/2} = 0$ almost everywhere implies $\mu = 0$ according to the uniqueness theorem of M. Riesz for the potentials of order α, cf. [12, § 10]). The logarithmic potential of this capacitary distribution λ has the following properties (cf. [7, Theorem 2.4]):

[1] By U^{μ}_{α} we denote the potential of order α of μ, that is, the potential of μ with respect to the kernel $|x-y|^{\alpha-n}$ of order α in R^n, $0 < \alpha < n$. If μ has a density f, that is, $d\mu = f(x)\,dx$, we may write U^{f}_{α} in place of U^{μ}_{α}.

(a) $U^\lambda \geq w(E)$ nearly everywhere[1] in E,

(b) $U^\lambda \leq w(E)$ everywhere in the support of λ.

For $n \leq 2$, the logarithmic kernel fulfills Frostman's maximum principle, and hence (a) and (b) may be replaced by: $U^\lambda = w(E)$ nearly everywhere in E, and $U^\lambda \leq w(E)$ everywhere in R^n, respectively. For $n \geq 3$ we have the following substitute for this latter inequality:

$$U^\lambda \leq w(E) + \log 2 \qquad \text{everywhere in } R^n \tag{5}$$

(cf. [8, § 2, formula (8)]). Likewise for arbitrary dimension n, the inequality $U^\lambda \geq w(E)$ holds everywhere in the interior of E. This may be proved in the manner devised by FROSTMAN [5, p. 37] for the potentials of order α.

It is well known (cf. e. g., [7, § 2.3]) that, for any bounded set $E \subset R^n$,

$$\gamma_*(E) = \sup_K \gamma_*(K) \qquad (K \text{ compact}, \; K \subset E). \tag{6}$$

The *exterior* logarithmic capacity $\gamma^*(E)$ of an arbitrary bounded set E is defined by

$$\gamma^*(E) = \inf_G \gamma_*(G) \qquad (G \text{ open}, \; G \supset E). \tag{7}$$

A bounded set E is called *capacitable* (with respect to the logarithmic kernel) if $\gamma^*(E) = \gamma_*(E)$. If E is capacitable, we may write simply $\gamma(E)$ for the logarithmic capacity $\gamma^*(E) = \gamma_*(E)$ of E. This is the case, in particular, if E is open or compact (cf. e. g., [7, p. 153 f.]).

3. A substitute for M. Riesz' composition formula and its applications

The following lemma coincides in essence with a formula stated in FROSTMAN [5, p. 61]. It serves as a substitute for the important composition formula of M. RIESZ for the kernels of order α in R^n:

$$\int_{R^n} |x-z|^{\alpha-n} |z-y|^{\beta-n} dz = c_{\alpha,\beta} |x-y|^{\alpha+\beta-n},$$

valid for $\alpha + \beta < n$. The formula to be discussed here corresponds to the limit case $\alpha + \beta = n$.

[1] The expression "nearly everywhere in E" means "everywhere in E except possibly in some set $N \subset E$ for which $\gamma_*(N) = 0$". Replacing γ_* by γ^*, we arrive at an analogous concept called "quasi-everywhere".

LEMMA 3.1. *Let A and B denote two concentric closed balls in R^n of radii* ϱ (fixed) *and* $R > 2\varrho$, *respectively. The function* $\psi(x, y; R)$ *of* $x \varepsilon A$ *and* $y \varepsilon A$ *defined by*

$$\frac{1}{\omega_n}\int_B |x-z|^{\alpha-n}|z-y|^{-\alpha}dz = \log\frac{R}{|x-y|} + \psi(x, y; R)$$

is continuous on $A \times A$ and approaches a certain constant c uniformly in $A \times A$ as $R \to +\infty$.

Proof. We may of course suppose that the common centre of A and B is the origin 0, and so $B = \{z \varepsilon R^n : |z| \le R\}$. We begin by studying the case $y = 0$. Introducing polar coordinates, whereby $dz = |z|^{n-1} d|z| dw$, we obtain for reasons of homogeneity, writing $t = |x|/|z|$,

$$\frac{1}{\omega_n}\int_B |x-z|^{\alpha-n}|z|^{-\alpha}dz = \int_{r/R}^{\infty} t^{-1}u_\alpha(t)\,dt,$$

where $r = |x|$, and where $u_\alpha(t)$ denotes the potential of order α of the uniform distribution of unit mass on the unit sphere in R^n, evaluated at a point of distance t from the origin. Clearly, $u_\alpha(t)$ is differentiable for $t > 1$ and for $0 \le t < 1$, and integrable over a neighbourhood of $t = 1$. Moreover,

$$u_\alpha(0) = 1; \quad u'_\alpha(0) = 0; \quad u_\alpha(t) = O(t^{\alpha-n})$$

as $t \to +\infty$. Hence the function v_α defined by

$$v_\alpha(t) = \begin{cases} t^{-1}u_\alpha(t) & \text{for } t > 1 \\ t^{-1}(u_\alpha(t) - 1) & \text{for } 0 < t < 1 \end{cases}$$

is bounded near 0 and integrable over $(0, +\infty)$. We now obtain

$$\frac{1}{\omega_n}\int_B |x-z|^{\alpha-n}|z|^{-\alpha}dz = \log\frac{R}{r} + V\left(\frac{r}{R}\right),$$

where $V(t) = \int_t^{\infty} v_\alpha(s)\,ds$ is continuous and approaches the limit

$$c = \int_0^{\infty} v_\alpha(t)\,dt$$

as $t \to 0$.—In the general case, let $|x| \le \varrho$, $|y| \le \varrho$, and $R > 2\varrho$. We compare the integral of $f(z) = |x-z|^{\alpha-n}|z-y|^{-\alpha}$ over $B = \{z \varepsilon R^n : |z| \le R\}$ with the integral of $f(z)$ over the ball $B' = \{z \varepsilon R^n : |z-y| \le R\}$ of centre y. Since the poles x and y belong to both balls (because $|x-y| \le 2\varrho < R$), the two integrals differ by a continuous function on $A \times A$, viz.

$$F(x, y; R) = \frac{1}{\omega_n}\int_{B-B'} f(z)\,dz - \frac{1}{\omega_n}\int_{B'-B} f(z)\,dz.$$

As the integrand f is $O(R^{-n})$, and the volumes of the sets $B-B'$ and $B'-B$ are $O(R^{n-1})$, we infer that $F(x, y; R)\to 0$ for $R\to\infty$, uniformly for $x\,\varepsilon A$, $y\,\varepsilon A$. Summing up, we obtain the representation

$$\frac{1}{\omega_n}\int_B |x-z|^{\alpha-n}|z-y|^{-\alpha}dz = \log\frac{R}{|x-y|} + V\left(\frac{|x-y|}{R}\right) + F(x, y; R),$$

from which the assertions of the lemma follow because

$$\psi(x, y; R) = V\left(\frac{|x-y|}{R}\right) + F(x, y; R).$$

It follows, in particular, from Lemma 3.1 that the logarithmic kernel $-\log|x-y|$ and the kernel

$$\Lambda_B(x, y) = \frac{1}{\omega_n}\int_B |x-z|^{\alpha-n}|z-y|^{-\alpha}dz$$

(considered on A) differ by the continuous function

$$\log R + \psi(x, y; R)$$

of $(x, y)\ \varepsilon A\times A$. We denote the supremum of the absolute value of this latter function over the compact set $A\times A$ by $M = M(R, \varrho)$. It depends on R, ϱ, α, and n. Taking, e. g., $R = 3\varrho$, we obtain a constant $M(3\varrho, \varrho)$ which we shall denote simply by $M(\varrho)$. Thus

$$\Lambda_B(x, y) - M(\varrho) \le \log\frac{1}{|x-y|} \le \Lambda_B(x, y) + M(\varrho) \tag{1}$$

for $x\,\varepsilon A$, $y\,\varepsilon A$. In particular, the class $\mathfrak{E}$ of measures μ, supported by A, whose energy is defined and finite, is the same for the two kernels $-\log|x-y|$ and $\Lambda_B(x, y)$. Since Λ_B is a definite kernel on A (cf. [7], § 3.5), we conclude that $\mathfrak{E}$ is a vector-space and that the class $\mathfrak{E}^+$ of positive measures in $\mathfrak{E}$ is a convex cone. Moreover, the logarithmic energy

$$\langle\mu, \mu\rangle = \iint \log\frac{1}{|x-y|}\,d\mu(x)\,d\mu(y)$$

is $\neq -\infty$. It is, in fact, $\geq -M(\varrho)\{\int|d\mu|\}^2$. This result will be improved considerably in § 4. Observe also that the logarithmic mutual energy $\langle\mu, \nu\rangle$ is defined and finite if $\mu, \nu\ \varepsilon\ \mathfrak{E}$, cf. [7, § 3.1].

The following two lemmas will not be used in the sequel. They are included on account of their role in [8, § 7]. The notations are those used in the preceding lemma (say, with $R = 3\varrho$). The characteristic function associated with a set E is denoted by φ_E. See also note 1, p. 5.

LEMMA 3.2. *Let* μ *denote a measure supported by the ball A, and put* $f = \varphi_B \cdot U^{\mu}_{n-\alpha}$. *Then the inequalities*

$$\frac{1}{\omega_n} U^f_\alpha - M(\varrho)\int |d\mu| \leq U^\mu \leq \frac{1}{\omega_n} U^f_\alpha + M(\varrho)\int |d\mu|$$

hold at any point of A at which the logarithmic potential U^μ *is defined* (hence everywhere in A if $\mu \geq 0$).

LEMMA 3.3. *Let* μ *denote a measure supported by A and of finite logarithmic energy* $\langle \mu, \mu \rangle$. *Then*

$$\left| \langle \mu, \mu \rangle - \frac{1}{\omega_n}\int_B [U^\mu_{n/2}]^2 dx \right| \leq M(\varrho)\left\{\int |d\mu|\right\}^2.$$

Each of these lemmas is derived from (1), or directly from Lemma 3.1, by integration with respect to $d\mu(y)$ (in Lemma 3.2) and $d\mu(x)d\mu(y)$ (in Lemma 3.3), followed by an application of Fubini's theorem. Thus we obtain

$$\frac{1}{\omega_n}\int_B [U^\mu_{n/2}(z)]^2 dz = \langle \mu, \mu \rangle + \iint [\log R + \psi(x, y; R)]\, d\mu(x)\, d\mu(y), \qquad (2)$$

from which Lemma 3.3 follows. Similarly in case of Lemma 3.2. In the special case where $\int d\mu = 0$ we arrive, following FROSTMAN [5, p. 61 f.], at the identity (2), § 2, when we let $R \to \infty$ in (2) under observation of the final assertion of Lemma 3.1.

4. The perfectness of the logarithmic kernel

LEMMA 4.1. *For any ball A of sufficiently small radius a, the restriction of the logarithmic kernel* $-\log|x-y|$ *to* $A \times A$ *is definite.*

Proof. Simple considerations of homogeneity will show that this property of the radius a is equivalent to the following inequality, valid for all measures (even of variable sign) of finite logarithmic energy, concentrated on some ball of arbitrary given radius ϱ:

$$\langle \mu, \mu \rangle \geq \log(a/\varrho)\cdot\left(\int d\mu\right)^2. \qquad (1)$$

Moreover it suffices to prove this inequality in the case $\varrho = 1$ of the unit ball B_1. The idea of the proof is classical in the case $n = 2$ (cf. De la Vallée-Poussin [14, § 47] and Deny [4, p. 164]. It consists in producing a measure λ with $\int d\lambda = 1$ whose logarithmic potential U^λ is constant, say $= L$, everywhere in the unit ball B_1. If μ denotes any measure of finite energy concentrated on B_1, and if we write $m = \int d\mu$, then $\mu - m\lambda$ is likewise of finite energy, and since its algebraic total mass is 0, its logarithmic energy is $\geqq 0$ according to (2), § 2. Evaluating this energy, we get

$$\langle \mu, \mu \rangle - 2\, m \int U^\lambda d\mu + m^2 \langle \lambda, \lambda \rangle \geqq 0,$$

and hence

$$\langle \mu, \mu \rangle \geqq (2L - \langle \lambda, \lambda \rangle)\, m^2. \tag{2}$$

The existence of a measure λ (of compact support) with the stated properties: U^λ = constant $(= L)$ on B_1, $\int d\lambda = 1$, can be proved as follows. For $n = 1$ or $n = 2$, the logarithmic kernel fulfills the maximum principle, and hence the capacitary distribution λ of unit mass on B_1 has the desired properties (cf. § 2), and we get $L = w(B_1) = \langle \lambda, \lambda \rangle$. This leads to the largest possible value a_n of a (in the case $n \leqq 2$): $a_n = \exp(w(B_1))$. For $n = 2$, λ is simply the uniform distribution of unit mass on the unit circle, and hence $w(B_1) = 0$ (= the value of U^λ at the centre 0). This gives $a_2 = 1$. For $n = 1$, it can be shown that λ has the density τ given by $\tau(x) = 0$ for $|x| > 1$ and

$$\tau(x) = \pi^{-1}(1 - x^2)^{-1/2} \quad \text{for} \quad |x| < 1;$$

and this leads to $w(B_1) = \log 2$, $a_1 = 2$ (cf. below).

For $n \geqq 3$, the capacitary distribution on the unit ball B_1 has no longer a constant logarithmic potential in B_1, and so the existence of a measure λ with constant U^λ in B_1 (and $\int d\lambda = 1$) must be verified in a different manner. Although it is possible to do this in an elementary way, we shall prefer to make use of the theory of distributions and at the same time determine explicitly the *best possible value* a_n of a. We propose to determine explicitly an *equilibrium distribution* T on the unit ball B_1 in R^n, that is, a distribution in the sense of Schwartz [13], supported by B_1, having the total integral $T(1) = 1$, and possessing a logarithmic potential which is constant on B_1. This equilibrium distribution T may then replace λ in the preceding argument in the case $n \leqq 2$ (in which case, actually, $T = \lambda$).[1]

[1] If, nevertheless, we insist upon constructing a *measure* λ with the desired properties, we merely have to "regularize" T by subjecting it to a homothetic transformation of R^n with respect to the origin and of a ratio $1 + r > 1$, followed by a convolution with some infinitely differentiable function $\varphi \geqq 0$, $\int \varphi(x)\, dx = 1$, supported by the ball of radius r about the origin. The logarithmic potential of the measure λ obtained in this manner has the constant value $\log\,[a_n/(1 + r)]$ in the unit ball B_1.

We begin by solving the corresponding problem for the potentials of order α in R^n instead of the logarithmic potential. For $0 < \alpha < 2$, the equilibrium distribution on B_1 is the positive measure T_α whose density τ_α is given by $\tau_\alpha(x) = 0$ for $|x| > 1$ and

$$\tau_\alpha(x) = \frac{2}{\omega_n} \frac{\Gamma(1 - \alpha/2 + n/2)}{\Gamma(1-\alpha/2)\Gamma(n/2)} (1 - |x|^2)^{-\alpha/2}$$

for $|x| < 1$. The constant value of the potential $U_\alpha^{T_\alpha}$ within B_1 coincides with the energy of order α of T_α. The common value is

$$u_\alpha = \frac{\Gamma(\alpha/2)\Gamma(1-\alpha/2 + n/2)}{\Gamma(n/2)}. \tag{3}$$

These results may be verified in the manner described in Pólya and Szegö [11] for the case $n \leq 3$ (cf. also M. Riesz [12, § 16] for the general case).—For an arbitrary value of α, the equilibrium distribution of order α on the unit ball B_1 in R^n can be obtained by analytic continuation of the above distribution T_α, and the constant value u_α of the potential $U_\alpha^{T_\alpha}$ within B_1 is given again by (3). (The "spectral measure" of B_1 is, therefore, $1/u_\alpha$; cf. Deny [4, p. 127].). For $\alpha = 2$ we find T_2 = the uniform distribution of unit mass on the unit sphere. For $\alpha > 2$, T_α is no longer a measure, but can be expressed as a "finite part" in the sense of Hadamard and Schwartz. (For $\alpha = 2k$, $k = 1, 2, \ldots$, T_α is a "multilayer" of order k on the unit sphere, cf. Deny [4, p. 129].)

Next we pass to the logarithmic potential by a differentiation with respect to the order α at $\alpha = n$ (cf. (1), § 2). If we apply the operator $-(\partial/\partial\alpha)_{\alpha=n}$ to both sides of the equation

$$U_\alpha^{T_\alpha} = u_\alpha \text{ in } B_1,$$

we obtain on the left the logarithmic potential of T_n. (The additional term is the total integral of $-(\partial T_\alpha/\partial\alpha)_{\alpha=n}$, and this vanishes because $T_\alpha(1) = 1$ for every α). The resulting equation

$$U^{T_n} = -\{\partial u_\alpha/\partial\alpha\}_{\alpha=n} \text{ in } B_1 \tag{4}$$

shows that $T = T_n$ *is the equilibrium distribution on the unit ball* B_1 in R^n, corresponding to the logarithmic kernel. Similarly, the logarithmic energy of $T = T_n$ is $-\{\partial u_\alpha/\partial\alpha\}_{\alpha=n}$. The largest possible value a_n of the radius a in Lemma 4.1 is now determined by

$$-\log a_n = \{\partial u_\alpha/\partial\alpha\}_{\alpha=n} = \frac{1}{2}\Psi(n/2) - \frac{1}{2}\Psi(1),$$

where $\Psi(t) = \Gamma'(t)/\Gamma(t)$. Explicitly,

$$\log\frac{1}{a_n} = \left\{\begin{array}{ll}(n-2)^{-1}+(n-4)^{-1}+\ldots+2^{-1} & \text{for even } n,\\ (n-2)^{-1}+(n-4)^{-1}+\ldots+1^{-1}-\log 2 & \text{for odd } n.\end{array}\right\} \quad (5)$$

THEOREM 4.1. *The logarithmic kernel* $-\log|x-y|$ *is perfect when considered on a closed ball* $A\subset R^n$ *of radius* $\varrho < a_n$.

Proof. The restriction of the logarithmic kernel to such a ball is *strictly definite* according to the inequality (1) together with the fact that the logarithmic energy of a measure of compact support is either finite or $+\infty$ (if at all defined), cf. § 3. Moreover, the logarithmic kernel in R^n is *regular* (i.e., satisfies the principle of continuity) by virtue of Kametani's theorem (cf. KUNUGUI [9, p. 78]); and so is therefore the restriction of $-\log|x-y|$ to $A\times A$. It follows from these two properties that this restriction is *consistent*, and hence perfect, cf. [7, Theorems 3.4.1 and 3.3].—Actually, the assertion of the theorem remains valid in the case $\varrho = a_n$ provided $n\geq 3$, because the sign of equality in (1) never occurs for any *measure* μ, but only for the equilibrium distribution T_n which is not a measure when $n\geq 3$.

In view of this perfectness of the logarithmic kernel (considered on A), the logarithmic potential of measures supported by A has all the properties described in [7, Chapter II]. First of all, we may introduce the interior and exterior *Wiener capacity* of arbitrary sets $E\subset A$:

$$\mathrm{cap}_* E = 1/w(E) = -1/\log\gamma_*(E), \quad (6)$$

$$\mathrm{cap}^* E = -1/\log\gamma^*(E). \quad (7)$$

Next, we may consider the (unique) interior and exterior *capacitary distributions* associated with an arbitrary set $E\subset A$, cf. [7, § 4]; and finally we may apply CHOQUET's theory of capacitability [2]. We prefer to state the results thus obtained in terms of the *logarithmic* capacity (instead of the Wiener capacity) and the capacitary distributions of *unit mass*. In this way we avoid the limitation to subsets of A; the extension to arbitrary bounded sets is simply a matter of applying a homothetic transformation, and using the fact that, for any constant $k>0$, the kernels $\log(k/|x-y|)$ and $\log(1/|x-y|)$ differ by the additive constant $\log k$.

THEOREM 4.2. *To any bounded set* $E\subset R^n$ *corresponds a unique measure* λ *with*

$$\int d\lambda = 1,\ \langle\lambda,\lambda\rangle = w(E) = -\log\gamma_*(E),$$

whose logarithmic potential has the following properties

(a) $U^{\lambda} \geqq w(E)$ nearly everywhere in E,

(b) $U^{\lambda} \leqq w(E)$ everywhere in the support of λ.

This measure λ is called the *interior capacitary distribution of unit mass* associated with E. There is a similar *exterior capacitary distribution* of unit mass, whereby $w(E)$ should be replaced by $w^*(E) = -\log \gamma^*(E)$, and the term "nearly everywhere" by "quasi-everywhere". If E is capacitable, these two capacitary distributions coincide. This is the case, in particular, if E is compact, in which case λ is supported by E and coincides with the capacitary distribution of unit mass on E discussed in § 2.—Returning to the two capacitary distributions associated with an arbitrary bounded set E, we finally observe that, as in § 2, properties (a) and (b) imply

$$U^{\lambda} \geqq w(E) \quad \text{everywhere in the interior of } E, \tag{8}$$

and

$$U^{\lambda} \leqq w(E) + \log 2 \quad \text{everywhere in } R^n. \tag{9}$$

Theorem 4.3. *If a bounded set $E \subset R^n$ is the union of an increasing sequence of sets E_p, then*

$$\gamma^*(E) = \lim_p \gamma^*(E_p).$$

This follows from Theorem 4.1 in view of [7, Theorem 4.4] in the case where E is contained in a ball A of radius $\varrho < a_n$. In the general case we apply first a suitable homothetic transformation as described above.—In the terminology of Choquet [2, § 15.3], this result means that the logarithmic capacity $\gamma(K)$ is *alternating* of order 1, a (when considered as defined on the class of all compact subsets K of, say, a fixed ball in R^n). Applying Choquet [2, § 30.2], we therefore obtain the following conclusion:

Theorem 4.4. *Every bounded analytic set* (in particular every bounded Borel set) *$E \subset R^n$ is capacitable with respect to the logarithmic kernel $-\log|x-y|$ in R^n:*

$$\gamma^*(E) = \gamma_*(E).$$

As mentioned in the introduction, the case $n = 2$ of Theorems 4.3 and 4.4 was settled by Choquet [3] even without the restrictions of boundedness. It is not known to the present author whether, for $n > 2$, these two theorems would subsist if the boundedness restrictions were dropped. (One could define $\gamma_*(E)$ and $\gamma^*(E)$ for arbitrary sets E by (6) and (7), § 2, respectively.)

References

[1]. H. Cartan, *Théorie du potentiel Newtonien.* Bull. Soc. Math. France, 73 (1945), 74—106.

[2]. G. Choquet, *Theory of capacities.* Ann. Inst. Fourier, 5 (1953—54), 131—295.

[3]. G. Choquet, *Capacitabilité en potentiel logarithmique.* Bull. Acad. Roy. Belgique, (5) 44 (1958), 321—326.

[4]. J. Deny, *Les potentiels d'énergie finie.* Acta Math., 82 (1950), 107—183.

[5]. O. Frostman, *Potentiel d'équilibre et capacité des ensembles.* Comm. Sém. Math. Lund, 3 (1935).

[6]. O. Frostman, *Potentiels de masses à somme algébrique nulle.* Kungl. Fysiogr. Sällsk. Lund Förh., 20 (1950), 1—21.

[7]. B. Fuglede, *On the theory of potentials in locally compact spaces.* Acta Math., 103 (1960), 139—215.

[8]. B. Fuglede, *On generalized potentials of functions in the Lebesgue classes.* To appear in Math. Scand., 8 (1960).

[9]. K. Kunugui, *Étude sur la théorie du potentiel généralisé.* Osaka Math. J., 2 (1950), 63—103.

[10]. M. Ohtsuka, *Sur un espace complet de mesures positives dans la théorie du potentiel.* Proc. Jap. Acad., 32 (1956), 311—313.

[11]. G. Pólya und G. Szegö, *Über den transfiniten Durchmesser* . . . J. reine u. angew. Math., 165 (1931), 4—49.

[12]. M. Riesz, *Intégrales de Riemann-Liouville et potentiels.* Acta sci. math. Szeged, 9 (1938), 1—42.

[13]. L. Schwartz, *Théorie des distributions.* Paris, 1950—51.

[14]. Ch. de la Vallée-Poussin, *Le potentiel logarithmique.* Paris-Louvain, 1949.

Indleveret til Selskabet den 8. september 1960.
Færdig fra trykkeriet den 31. december 1960.